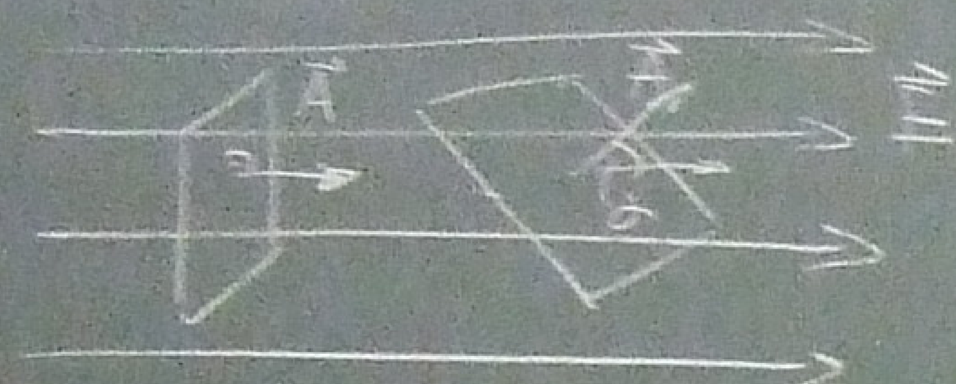
Das Gauß'sche Gesetz

Fluß ϕ des elektrischen Feldes:

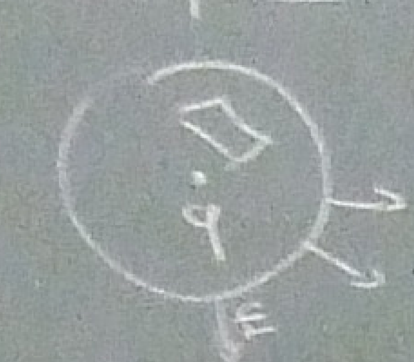


(a) $\phi = \vec{A}\cdot\vec{E} = AE$

(b) $\phi = AE\cdot\cos\alpha$

Fläche $\vec{A}$ steht $\perp$ zur Fläche; $|\vec{A}| = A$

Beispiel



Kugeloberfläche mit Ladung q im Mittelpunkt $\vec{A} \perp \vec{E}$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad A = 4\pi r^2$$

$$\phi = \iint_A \vec{E}\,d\vec{A} = \frac{q}{\varepsilon_0}$$

Gilt auch für beliebige Flächen

Der Fluß des elektrischen Feldes aus einer beliebigen, geschlossenen Fläche ist die Ladung q/ε_0 im Inneren. (Gauß'scher Satz)

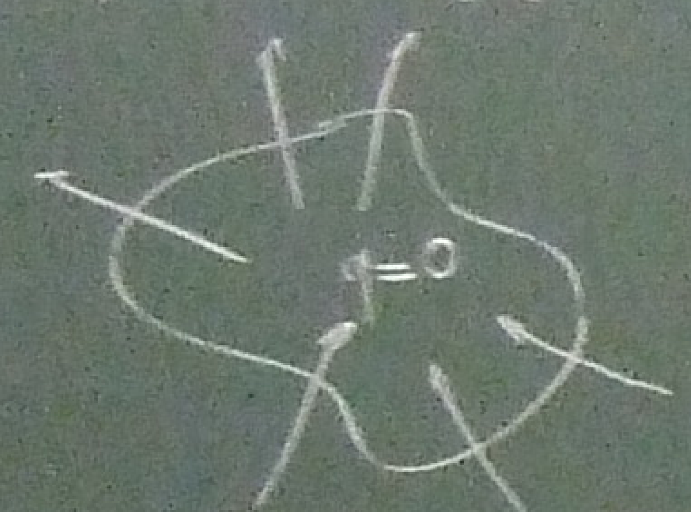
Mathematisch:

$$\phi = \oiint_A \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV = q/\varepsilon_0$$

Bemerkungen:

$\phi = 0 \rightsquigarrow$ keine Ladung im Inneren.

- Gilt auch für bewegte Ladungen
- Analogie: Saujagd



- kein neues Gesetz, sondern Folgerung aus Coulombgesetz: $F \sim 1/r^2$

Integralsatz von Gauß

Für ein beliebiges Vektorfeld gilt:

$$\oiint_A \vec{E}\,d\vec{A} = \iiint_V \operatorname{div}\vec{E}\,dV \overset{!}{=} \iiint \frac{\rho}{\varepsilon_0}\,dV$$

Deshalb: $\boxed{\operatorname{div}\vec{E} = \rho/\varepsilon_0}$

$$\operatorname{div}\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Dies ist das 1. Maxwell'sche Gesetz in differentieller Form.

Poisson-Gleichung

$$\vec{E} = -\operatorname{grad}\varphi$$

$$\operatorname{div}\vec{E} = -\operatorname{div}\operatorname{grad}\varphi = -\Delta\varphi \overset{!}{=} \rho/\varepsilon_0$$

$$\boxed{\Delta\varphi = -\rho/\varepsilon_0}$$

Laplace-Operator: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

„Krümmung von φ ist prop. der Ladungsdichte ρ"

Integralsatz von Stokes

$$\oint_C \vec{E}\,d\vec{r} = \iint_A \operatorname{rot}\vec{E}\,d\vec{A} = 0$$

$\vec{E}$ = elektr. Feld

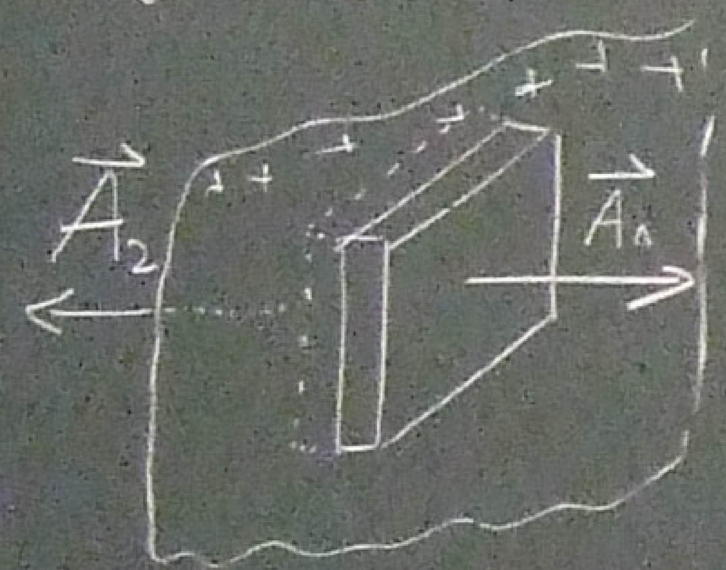
Umlaufintegral von $\vec{E}$ längs einer geschlossenen Kurve C ist gleich dem Fluß von rot $\vec{E}$ durch die von C geschlossene Fläche.

wegen $\vec{E} = -\operatorname{grad}\varphi$: $\boxed{\operatorname{rot}\vec{E} = 0}$

Später: $\operatorname{rot}\vec{E} = -\frac{\partial B}{\partial t}$

Anwendung Gaußsatz

Sprung des E-Feldes an einer ausgedehnten Ladungsschicht:



Dünne Schachtel

$$\oiint \vec E \, d\vec A = \iiint \rho \, dV$$

Nur der Betrag der Flächen A_1, A_2 trägt bei dünner Schachtel bei:

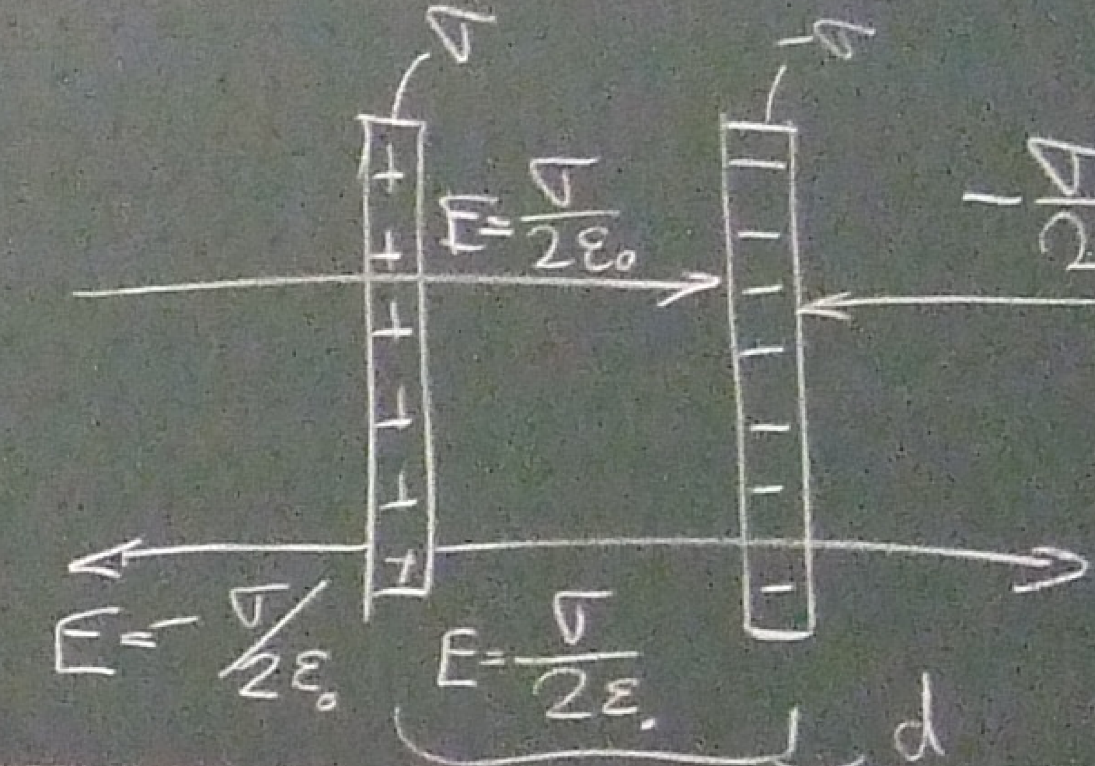
$$\vec A_1 = -\vec A_2$$

$$\oiint \vec E \cdot d\vec A = \vec A_1 \cdot \vec E_1 + \vec A_2 \cdot \vec E_2 = \left(E_1^\perp - E_2^\perp\right) A = \frac{q}{\varepsilon_0}$$

$$\rightsquigarrow \boxed{\Delta E_\perp = \frac{\sigma}{\varepsilon_0}}$$

$\sigma = q/A$ Flächenladungsdichte

Beispiel Plattenkondensator



Σ $E = 0$ $\boxed{E = \frac{\sigma}{\varepsilon_0}}$ $E = 0$

Potentialdifferenz bei Abstand d

$$U = (-)\int_0^d \vec E \cdot d\vec s = E \cdot d = \frac{\sigma}{\varepsilon_0} d = \frac{q\, d}{A \varepsilon_0} \sim q$$

$E = \text{const}$

Allgemein: $U \sim q$

Kapazität C: $Q = C\,U$

C wird gemessen in Farad $F = \frac{\text{Ladung}}{\text{Spannung}} = \frac{C}{V}$

Bemerkung:

Typische Werte sind $\mu F \ldots pF$

(Fahrradstandlichter: F)

lrz, docweb

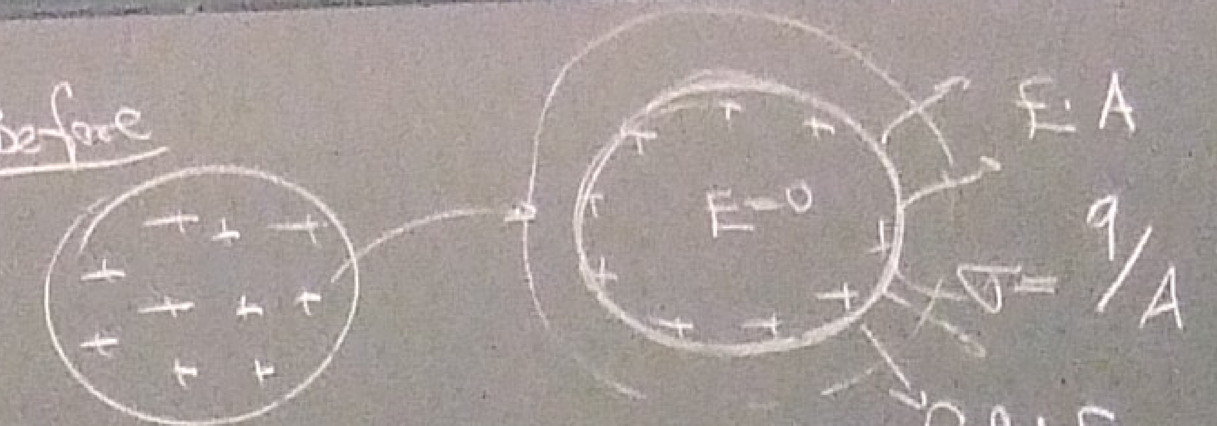
Conductor in an electrical field

Charges are free to move in a conductor.

⇝ They move in the electrical field until no forces are excerted on them.

Example: Charges in a conductor move to its surface.

Before

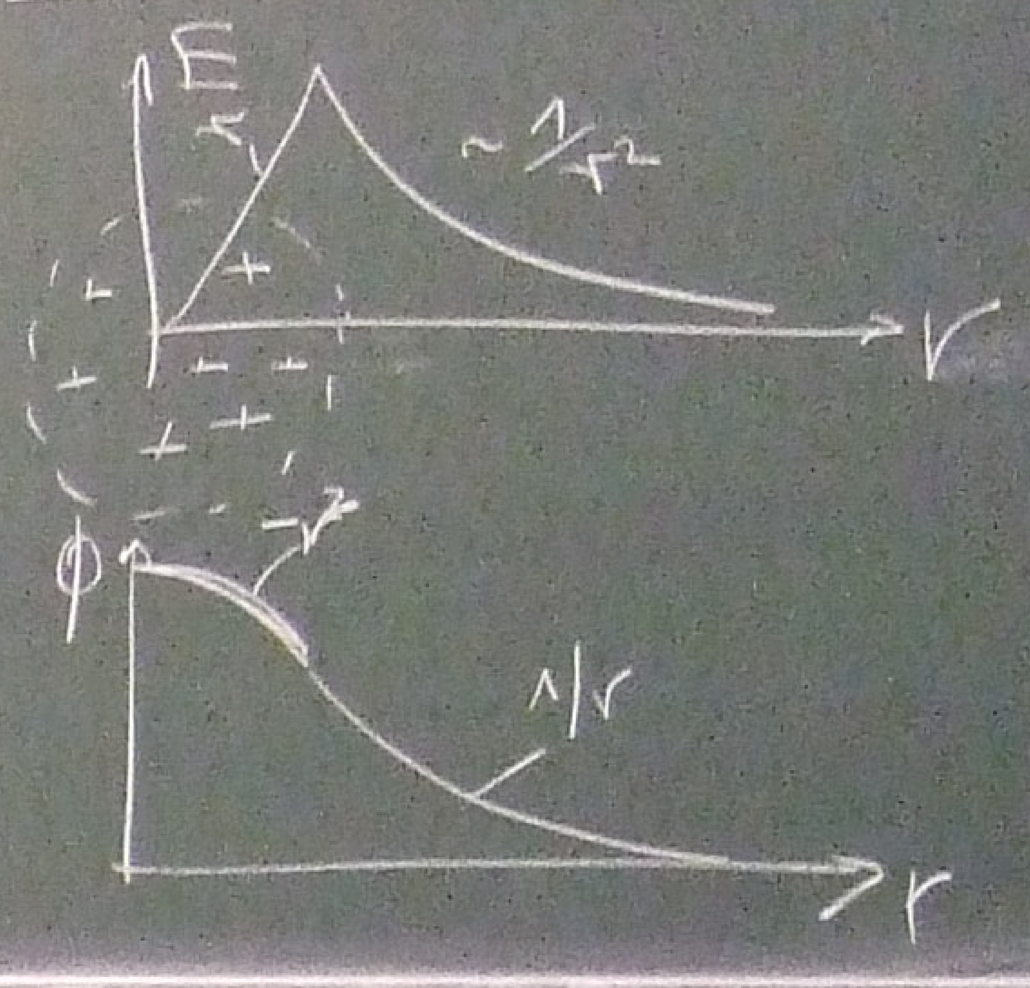


Inside the conductor the field E and the charge density ρ are zero.

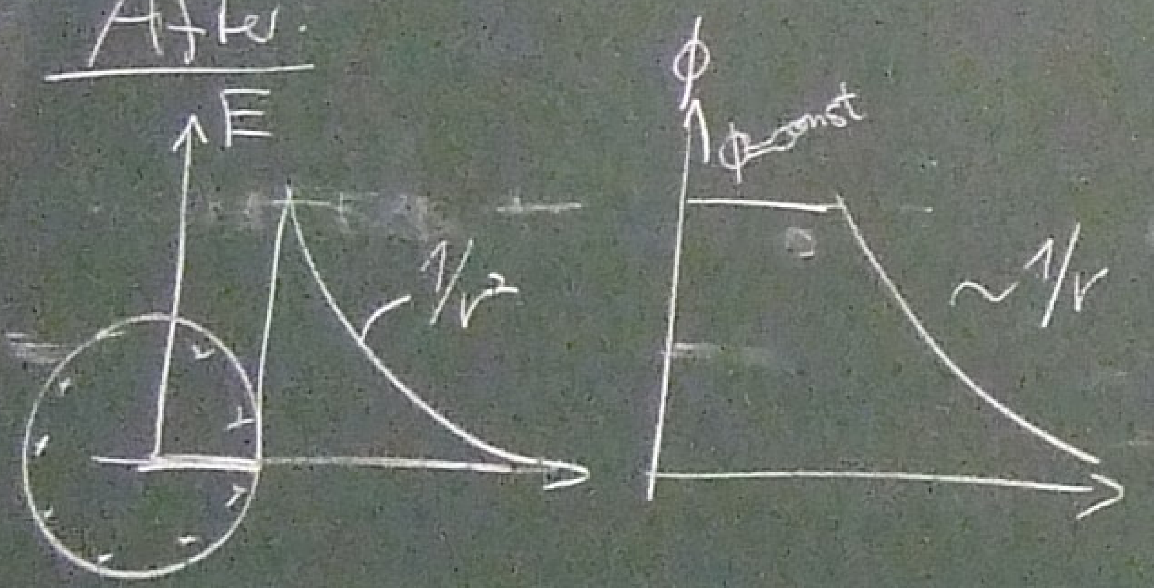
⇝ From the charge density (area) $\sigma = q/A$ we obtain the outside field:

$$E_\perp^{outside} = \sigma/\varepsilon_0$$

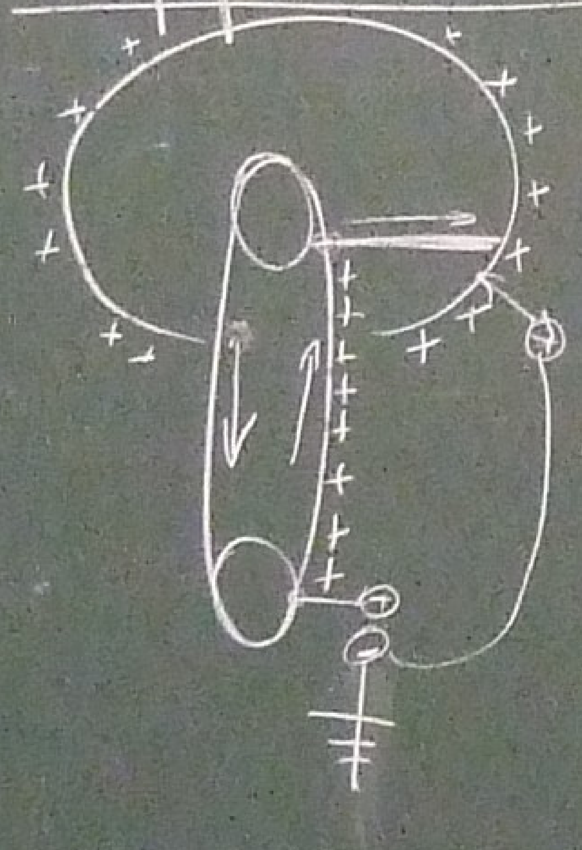
Before:



After:



Applications: Von de Graaff-Generator



Charge the sphere from the inside.

⇝ High Voltages (10 MV)

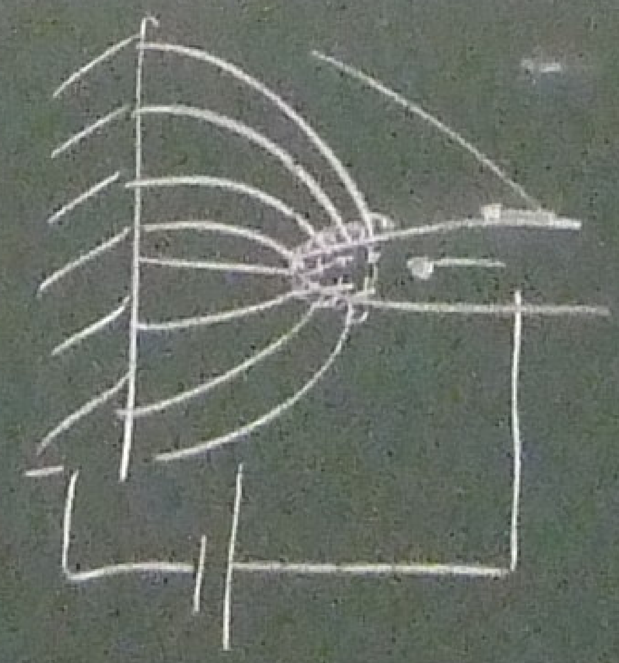
Elestatic Discharge

Discharge in air depends on the electrical field, i.e. the forces necessary to create a plasma

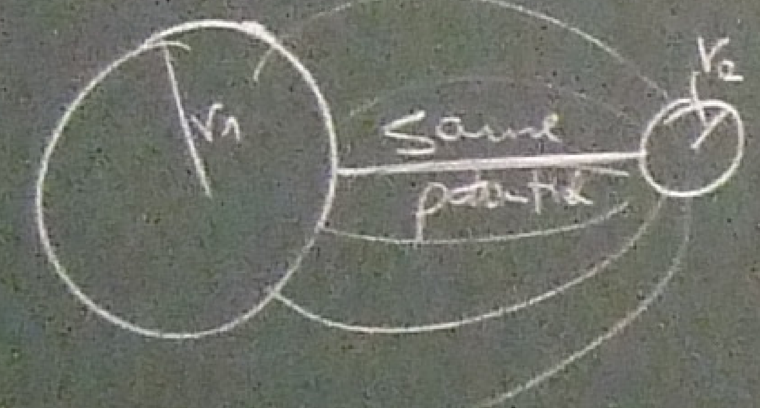
$$E_{max}^{air} \approx 3 \cdot 10^6 \frac{V}{m}$$

Just prior to discharge: St. Elmo's fire (Elmsfeuer)

Electrical field is high at sharp tips (of a conductor)



Simplest Model



Charged spheres connected by conductor.

Same potential

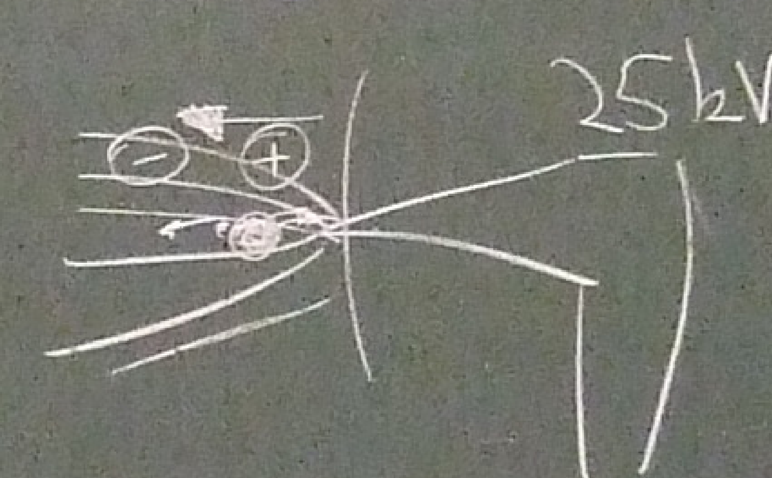
$$\frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r_1} \overset{!}{=} \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r_2} \quad (1)$$

Electrical field @ surface:

$$\sigma = \frac{Q}{4\pi r^2} = \varepsilon_0 E$$

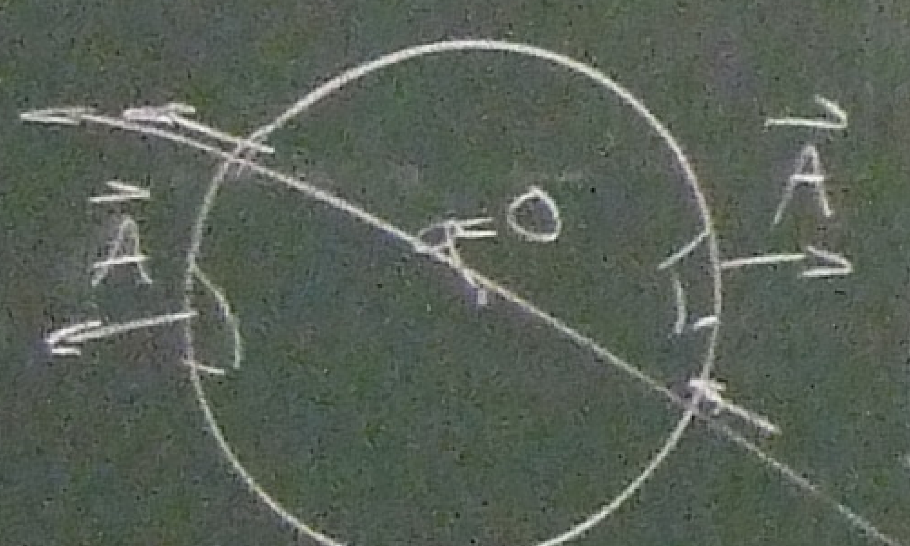
$$\frac{E_1}{E_2} = \frac{\overset{\sim r_1}{Q_1}\, r_2^2}{r_1^2\, \underset{\sim r_2}{Q_2}} \overset{(1)}{=} \frac{r_2}{r_1}$$

⇝ Small radius ⇝ large field



Faraday Cage

Even in external fields, $E = 0$ inside a hollow conductor.



$$\oiint \vec{E}\, d\vec{A} = 0$$

Reason: Poisson equation: $\Delta\varphi = -\rho/\varepsilon_0$

Inside the cage: $\Delta\varphi = 0$ (8)

$[\not\Rightarrow E = -\nabla\varphi = 0]$

Boundary condition due to conductor: $\varphi = \text{const}$ at boundary

(Equipotential)

Mathematics

Boundary condition $\varphi = \text{const}$ imposes $\varphi = \text{const}$ inside ⇝ $E = 0$ inside.